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Turbiner's conjecture in three dimensions

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Abstract

We prove a modified version of Turbiner's conjecture in three dimensions and we give a counter-example to the original conjecture. The Lie algebraic Schrödinger operators corresponding to flat metrics of a certain restricted type are shown to separate partially in Cartesian or cylindrical or spherical coordinates.

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1. Introduction

The aim of this article is to extend results related to separation of variables for flat Lie algebraic Schrödinger operators. Originally, in [11], Alexander Turbiner conjectured the following:

Conjecture 1 (*Turbiner*). *In $\mathbb{R}^2$ there exist no quasi-exactly solvable or exactly solvable problems containing the Laplace–Beltrami operator with flat-space metric tensor, which are characterized by non-separable variables.*

This conjecture was reformulated in more geometrical terms by Rob Milson. The conjecture, which the work in this paper is based on, now reads as follows.

Conjecture 2 (*Turbiner, Second Version*). *Let H be a Lie algebraic Schrödinger operator defined on a two-dimensional manifold. If the symbol of H engenders a Euclidean geometry, i.e. if the corresponding Gaussian curvature is zero, then the spectral equation $H\psi = E\psi$ can be solved by a separation of variables.*

This conjecture is false in general. A counter-example is given by Rob Milson in [6] and [7], together with a proof of a modified version of the conjecture. By adding two extra assumptions, namely an imprimitive action and a compactness requirement, one can prove that the spectral equation can be solved by separation of variables. Furthermore, the imprimitivity hypothesis implies even more than expected: separation will occur in either a Cartesian or a polar coordinate system.

In this paper, it is shown that, in three dimensions, the original conjecture is also false and a proof of a modified version of the 3D-Turbiner's conjecture is given. Again the compactness requirement is indispensable, a condition

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related to the imprimitivity of the action is necessary and a third condition, related to the contravariant metric, will be imposed. In three dimensions, the invariant foliation of an imprimitive action can be a family of curves or a family of surfaces. In the proof of our result, the leaves of the foliation need to be surfaces and such an imprimitive action will be called 2-imprimitive. Like for the two-dimensional case, the 2-imprimitivity of the action will ensure that separation, here only partial, will occur in a Cartesian, cylindrical or spherical coordinate system.

The proofs of both modified versions of the conjecture are based on the following ideas. First, the imprimitive action, which is 2-imprimitive in three dimensions, induces an invariant foliation Λ, for which the leaves are hypersurfaces and will be denoted by $\{\lambda = \text{constant}\}$. The Schrödinger operator $\mathcal{H}$ is Lie algebraic; thus, it is an element of the enveloping algebra of a Lie algebra of first-order differential operators. When applied to λ, the elements of the generating Lie algebra must give back functions of λ. The operator $\mathcal{H}$ will enjoy the same property, that is $\mathcal{H}(\lambda) = f(\lambda)$. Combining the fact that the operator is Lie algebraic with the imprimitivity of the action, one can prove that the leaves of $\Lambda^{\perp}$, the foliation which is perpendicular to Λ, are necessarily geodesics. Then, one of the key tools, the Tiling Theorem, gives a global map from a Euclidean space to our manifold. Thus on pulling back the operator, the leaves of the perpendicular foliation are straight lines. Note that the Tiling Theorem follows from an intermediate one: the Trapping Theorem.

In this setting, one can show that the invariant leaves can only be prescribed curves or surfaces. In two dimensions the curves need to be straight lines or concentric circles, while in three dimensions the surfaces have to be planes, cylinders or spheres. In this context, λ will be either a Cartesian or a radial coordinate. Finally, using the appropriate coordinate system, one checks that the equation $\mathcal{H}\psi = E\psi$ separates with respect to the coordinate λ.

Despite the fact that the path followed to prove the modified 3D-Turbiner's conjecture is similar to the one given in [7], there are several important issues, which were absent in the two-dimensional case, and which appear in our study. We had first to select the appropriate flatness criteria, while in two dimensions there is no such choice. In order to prove the 3D-Trapping Theorem, one has to impose that the diagonal terms of the Ricci curvature tensor be zero. For the 3D-Tiling Theorem, it is the Riemannian curvature tensor that needs to vanish. For this proof of the 3D-Trapping and Tiling Theorems, we have to assume that either the metric can be diagonalized, or that **M** is a transverse, type changing manifold. For the first case, to conclude both theorems, an extra requirement of genericity of the contravariant metric needs to be added. This requirement is related to the non-invertible factors of its components and will be defined later. For the transverse, type changing manifold, we will see that such metric can always be diagonalized and is necessarily generic.

The determination of the possible foliations requires a different approach. While in two dimensions one could only consider the possible foliations by straight lines to conclude, in three dimensions one has to keep in mind the entire picture of the two perpendicular foliations in order to determine the three types of leaves. The arguments are not sophisticated but the proof is long enough for us to devote an entire section to it. Another distinction from the two-dimensional case is the fact that separation of variables is only partial. Indeed, as in the work of Rob Milson, one can isolate one variable, λ, but we are left, in three dimensions, with two variables for which nothing can be said.

In Section 2, we briefly describe the context of Turbiner's conjecture. We define all the notions employed in this paper and we give an example that illustrates how the imprimitivity of the action induces separation of variables. Section 3 fills in the gaps needed to generalize the proof of both 3D-Tiling and 3D-Trapping Theorems. The proofs of these two theorems are omitted since, once this work done, both generalizations are straightforward. In Section 4, we show that, after pulling back the metric to $\mathbb{R}^3$, the only possible leaves of the foliation are planes, cylinders and spheres. This fourth section, involving a succession of simple ad hoc arguments, is crucial for its consequences although the proof itself may be skipped at first reading. Using the results exhibited in the two preceding sections, Section 5 is devoted to the proof of the 3D modified conjecture. Finally a counter-example to the three-dimensional general form of Turbiner's conjecture is exhibited in Section 6.

We conclude by noting that there are deep connections between separation of variables, exact solvability and superintegrability; read for instance [9,3]. However, these lie outside the scope of our paper.

2. General setting

In this section, we introduce the framework and the notions necessary to prove the three-dimensional version of the modified Turbiner's conjecture. The two-dimensional version, see [7,6] for a complete proof, is also discussed.

Recall that a Schrödinger operator on a n-dimensional Riemannian manifold is a second-order differential operator of the form

$$\mathcal{H} := -\frac{1}{2}\Delta + U,$$

where Δ is the Laplace–Beltrami operator and U is the potential function for the physical system under consideration. If $g^{(ij)}$ is the contravariant metric in a local coordinate chart and g its determinant, the operator is given by

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j=1}^{n}\left[g^{ij}\partial_{ij} + \partial_i(g^{ij})\partial_j - \frac{g^{ij}\partial_i(g)}{2g}\partial_j\right] + U.$$

The main purpose of this paper is to study separation of variables for operators that are equivalent to Schrödinger operators. To this end we want to consider an adequate notion of equivalence, one that preserves the formal spectral properties of the operators under consideration. The appropriate notion, which will be used throughout our work, is the following. Two differential operators are *locally equivalent* if there is a gauge transformation $\mathcal{H} \to e^{\lambda}\mathcal{H}e^{-\lambda}$ and a change of variable that relate one to the other. With this notion of equivalence in hand, one can, in principle, verify whether a general second-order differential operator $\mathcal{H}_0$ is equivalent to a Schrödinger operator. Indeed, every second-order differential operator can be given locally by

$$\mathcal{H}_0 = -\frac{1}{2}\sum_{i,j=1}^{n} g^{ij}\partial_{ij} + \sum_{i}^{n} h^i\partial_i + U.$$

If the contravariant tensor $g^{(ij)}$ is non-degenerate, that is if g does not vanish, the operator can be expressed as

$$\mathcal{H}_0 = -\frac{1}{2}\Delta + \vec{V} + U, \tag{2.1}$$

where $\vec{V}$ is a vector field that has to be eliminated by a further transformation, if possible. In this context, the equivalence condition can be restated as follows: $\mathcal{H}_0$ is gauge equivalent to a Schrödinger operator if and only if $\vec{V}$ is a gradient vector field, with respect to the metric $g^{(ij)}$. For obvious reasons, this criterion is called the *closure condition* and if $\vec{V} = \nabla(\lambda)$, the gauge factor has to be $e^{\frac{\lambda}{2}}$.

A differential operator $\mathcal{H}_0$ is *Lie algebraic* if it is an element of the universal enveloping algebra of a finite dimensional Lie algebra of first-order differential operators. Given a representation of a finite dimensional Lie algebra $\mathfrak{g}$ by vector fields on a manifold $\mathbf{M}$, one constructs a representation by first-order operators in the following way; see [2] for details. For each element $a \in \mathfrak{g}$, we define a first-order differential operator

$$T_a = a^{\pi} + \eta(a),$$

where a^{π} is a vector field on $\mathbf{M}$ and η is an element of $H^1(\mathfrak{g}, \mathcal{C}^{\infty}(\mathbf{M}))$. Thus for $\{a_1, \ldots a_m\}$, a basis for $\mathfrak{g}$, a second-order Lie algebraic differential operator is given by

$$\mathcal{H}_0 = \sum_{i,j=1}^{m} C^{ij}T_{a_i}T_{a_j} + \sum_{k=1}^{m} L^k T_{a_k}, \tag{2.2}$$

where C^{ab} and L^c are real numbers and without loss of generality $C^{ab} = C^{ba}$. The class of Lie algebraic operators is closed under gauge transformation; hence it makes sense to consider Lie algebraic operators that are equivalent to Schrödinger operators. Again, if the induced contravariant metric $g^{(ij)}$ is non-degenerate, the closure condition can be easily verified. This condition can be written in terms of the initial coefficients C^{ab} and L^c; however this leads to complicated PDE's.

We now give an example to better explain these notions. Later on, this same example will be used to illustrate the conjecture. Consider $\mathbf{M}$ a three-dimensional manifold diffeomorphic to $\mathbb{R}^3$ and the Lie algebra $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$ spanned by the first-order differential operators

$$T_1 = \partial_u, \qquad T_2 = u\partial_u, \qquad T_3 = \partial_v, \qquad T_4 = v\partial_v, \qquad T_5 = \partial_w, \qquad T_6 = w\partial_w.$$

We define a Lie algebraic operator $\mathcal{H}_0$ with the following choice of coefficients:

$$C^{ab} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad L^c = \begin{pmatrix} 0 \\ 2\alpha \\ 4\beta - 4 \\ 4\alpha \\ 4\beta + 4\gamma - 6 \\ 4\alpha \end{pmatrix},$$

where α, β and γ are real numbers. In terms of the (u, v, w) coordinates, the operator reads as follows:

$$\mathcal{H}_0 = \frac{1}{2}\partial_{uu} + 2u\partial_{uv} + 2u\partial_{uw} + 2v\partial_{vv} + 4v\partial_{vw} + 2w\partial_{ww} + 2\alpha u\partial_u + (4\beta - 3 + 4\alpha v)\partial_v + (4\beta + 4\gamma - 5 + 4\alpha w)\partial_w.$$

The metric associated with this operator is

$$g^{(ij)} = -\begin{pmatrix} 1 & 2u & 2u \\ 2u & 4v & 4v \\ 2u & 4v & 4w \end{pmatrix}. \tag{2.3}$$

If we forget the degeneracy issue for a moment, we can rewrite the operator $\mathcal{H}_0$ in terms of the Laplace–Beltrami operator associated with the metric (2.3). We obtain

$$\mathcal{H}_0 = -\frac{1}{2}\Delta + 2\alpha u\partial_u + (4\beta + 4\alpha v)\partial_v + (4\beta + 4\gamma + 4\alpha w)\partial_w.$$

A direct calculation shows that the undesirable first-order term can be expressed as the gradient, with respect to (2.3), of the scalar function

$$\lambda = \alpha w + \beta \ln|v - u^2| + \gamma \ln|w - v|.$$

The closure condition is then satisfied and, scaling with the factor $e^{\frac{\lambda}{2}}$, the operator constructed is gauge equivalent to the following Schrödinger operator:

$$\mathcal{H} = -\frac{1}{2}\Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2 w + \frac{\beta(\beta - 1)}{v - u^2} + \frac{\gamma(\gamma - 1)}{w - v}.$$

Observe that the tensor $g^{(ij)}$ fails to be of full rank when $g = 16(w - v)(u^2 - v) = 0$. The inverse tensor $g_{(ij)}$ is singular on the sets $\{w = v\}$ and $\{v = u^2\}$; thus the inner product is not defined everywhere. We therefore have to allow degeneracy for the contravariant metric. However, despite this flexibility, we want the metric to behave reasonably well on the degeneracy locus. For this reason, we introduce a generalization of the pseudo-Riemannian structure.

For $\mathbf{M}$ a real, analytic manifold and $g^{(ij)}$ a type $(2, 0)$ tensor field, we denote as $\mathbf{D}_g$ the locus of degeneracy of the tensor. The analyticity requirement implies that $\mathbf{D}_g$ is empty or a codimension 1 subvariety or $\mathbf{M}$. We set $\mathbf{M}_0 = \mathbf{M} \setminus \mathbf{D}_g$ and we assume that $g^{(ij)}$ is not identically degenerate. Thus $\mathbf{M}_0$ is an open, dense subset of $\mathbf{M}$ and the connected components of $\mathbf{M}_0$ are pseudo-Riemannian manifolds with boundary in $\mathbf{D}_g$.

The pair $(\mathbf{M}, g^{(ij)})$ is called an *almost-Riemannian manifold* if for all pairs u, v of analytic vector fields with non-degenerate plane section $u \wedge v$ on $\mathbf{M}_0$, the sectional curvature function $K(u \wedge v)$ has removable singularities on $\mathbf{D}_g$. Remark that if the sectional curvature is constant on connected components, which is the case if the Riemannian curvature is zero, then $\mathbf{M}$ is an almost-Riemannian manifold.

Throughout this paper, we will focus on components of $\mathbf{M}_0$ for which the metric is positive definite and for which the Riemannian curvature tensor is zero. For instance, in the previous example, $g^{(ij)}$ is positive definite on $\mathbf{R} = \{(u, v, w) | u^2 < v < w\}$ and we can easily check that the Riemannian curvature vanishes identically.

We will see that separation of variables is, in our context, closely related to foliations by geodesics. Thus we would prefer to work in Euclidean geometry, where the geodesics are straight lines, instead of working on $\mathbf{M}$, a flat analytic manifold. The 3D-Tiling Theorem will help us to achieve this by showing that, under certain conditions, there exists a global real–analytic map from $\mathbb{R}^3$ to $\mathbf{R} \subset \mathbf{M}$ where the contravariant metric of $\mathbf{R}$ is the pushforward of the Euclidean metric. The following definitions and propositions will be necessary to establish this theorem.

To better understand the overall behavior of the manifold around the degenerate points, we need to quantify the degeneracy of the contravariant metric. The degenerate points can be broken up into two categories. A point $p \in \mathbf{D}_g$ is called *unreachable* if all smooth curves with end points p have infinite length in the metric $g_{(ij)}$. Conversely a degenerate point is called *reachable* if it can be attained by a finite length curve. If $\gamma(t) : (0, 1) \to \mathbf{M}_0$ is a geodesic segment, we denote as T the largest number, possibly ∞, such that $\gamma(t)$ can be extended by a geodesic with domain $(0, T)$. For $\mathbf{R}$ an open connected component of $\mathbf{M}_0$, we say that $\mathbf{M}$ is *complete within* $\mathbf{R}$ whenever for all geodesic segments lying within $\mathbf{R}$, either $T = \infty$, or $\lim_{t\to T} \gamma(t)$ is a reachable boundary point of $\mathbf{R}$. This extends the notion of completeness to almost-Riemannian manifolds. The following proposition will be useful.

Proposition 2.1. *Suppose the signature of $g^{(ij)}$ is positive definite within $\mathbf{R}$, and that $\mathbf{R}$ is contained in a compact subset of $\mathbf{M}$. Then $\mathbf{M}$ is complete within $\mathbf{R}$.*

The degenerate points are given by the zero set of the determinant of a 3 by 3 matrix which is, in general, not easy to handle. To circumvent this issue, we will assume that either

(1) $g^{(ij)}$ can be diagonalized, or
(2) $\mathbf{M}$ is a transverse, type changing manifold.

For the case (1), there exists locally a coordinate system for which $g^{(ij)}$ is expressed as

$$g^{(ij)} = \begin{pmatrix} P(x, y, z) & 0 & 0 \\ 0 & Q(x, y, z) & 0 \\ 0 & 0 & R(x, y, z) \end{pmatrix}. \tag{2.4}$$

Thus, the determinant is given by the simple equation $g = PQR$ and we can assume without loss of generality that the metric is degenerate at the origin. We define the *order* of an analytic function to be the smallest total degree of all the monomials with a non-zero coefficient in its Taylor development. Thus, the order of g will be the sum of the orders of the diagonal components of (2.4). Note that the smaller the order of g is, the closer the metric is to being non-degenerate at the origin.

The next requirement will be needed to prove the three-dimensional version of the Trapping and Tiling Theorems when the metric is diagonal. This condition does not appear in the two-dimensional case and we do not know yet if it is necessary. We say that a contravariant metric tensor $g^{(ij)}$ given as (2.4) is *generic* if the components of the diagonal do not share non-invertible factors. For instance the metric

$$g^{(ij)} = \begin{pmatrix} (1+x)^2 & 0 & 0 \\ 0 & (1+x)y & 0 \\ 0 & 0 & xz \end{pmatrix}$$

is generic while the metric

$$g^{(ij)} = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & xy & 0 \\ 0 & 0 & xz \end{pmatrix}$$

is not.

For the case (2), it can be deduced that the metric can be diagonalized and its diagonal form is generic. Indeed, recall that $\mathbf{M}$ is a transverse, type changing analytic m-dimensional manifold if $\mathbf{M}$ is an analytic manifold with a contravariant metric $g^{(ij)}$ such that at any point x in the degenerate locus $\mathbf{D}_g$, we have:

(1) $\mathrm{d}(det(g^{(ij)}))\,|_x \neq 0$ for some (and hence any) coordinate system,
(2) $Rad_x := \{v_x \in T_x^*\mathbf{M} \mid g^{(ij)}(v, \cdot) = 0\}$ is transverse to $T_x^*\mathbf{D}_g$,

One can show, see [4] for details, that around any degenerate point, there exist local natural coordinates $\{x^1, \ldots, x^m\}$ such that

$$g^{(ij)} = \begin{pmatrix} g^{(ab)} & 0 \\ 0 & x^m \end{pmatrix},$$

where $g^{(ab)}$ is non-degenerate. In the three-dimensional case, $g^{(ab)}$ is a two by two matrix and can be diagonalized into invertible functions. This leads us to a contravariant metric

$$g^{(ij)} = \begin{pmatrix} P(x,y,z) & 0 & 0 \\ 0 & Q(x,y,z) & 0 \\ 0 & 0 & z \end{pmatrix},$$

for which the genericity property is satisfied.

Once the 3D-Tiling Theorem have been established, there will be one last major property needed in our study: the imprimitivity of the action. Since we are dealing with a Lie algebraic operator, we can assume that the domain of the operator is a homogeneous space $\mathbf{M} = \mathbf{G}/\mathbf{H}$ where $\mathfrak{g}$ is the Lie algebra corresponding to $\mathbf{G}$. Recall that the action of $\mathbf{G}$ on $\mathbf{M}$ is *imprimitive* if there exists a foliation of $\mathbf{M}$ that is invariant under the action of $\mathbf{G}$. In three-dimensional Euclidean space, the invariant leaves can be either curves or surfaces; see [5] for a more detailed description of the possible leaves. Throughout this paper we will only consider foliations by surfaces; this type of action will be called *2-imprimitive action*. Note however that it would be very interesting to study the case of a foliation by curves. In infinitesimal terms, if the leaves of the foliation are given by $\{\lambda = \text{constant}\}$, then $a^{\pi}(\lambda) = f(\lambda)$ for every element of the Lie algebra $\mathfrak{g}$. This second criterion can be generalized to extend the notion of 2-imprimitivity to differential operators. If the level sets of the function λ are the leaves of the foliation, the operator T_α is said to act on 2-imprimitively if $T_\alpha(\lambda)$ and λ are functionally dependent. One can easily show, see [7] for details, the following.

Proposition 2.2. *If the operators $\{T_a : a \in g\}$ act 2–2-imprimitively, then there is a* **G***-invariant foliation by surfaces on* **M**.

The central point here is that Lie algebraic operators generated by these 2-imprimitive generators will behave in the same way. Indeed the operator $\mathcal{H}_0$ applied to λ will give back a function of λ. On taking λ as coordinate, the operator will separate in that variable and we will show that the equivalent Schrödinger operator $\mathcal{H}$ will also separate partially.

One of the key arguments for the final theorem is that the invariant foliation Λ is perpendicular to a geodesic foliation. This is due to the Lie algebraic construction of the metric $g^{(ij)}$, and, according to the 3D-Tiling Theorem, this perpendicular foliation can be pulled back to the Euclidean space where the geodesics are well known: straight lines. We will not go into the details, everything being already exhibited in [7] and [6], but we will state the main results necessary to prove the theorem.

We will denote as $\Lambda^{\perp}$ the distribution of tangent vectors that are perpendicular to Λ. For a Lie algebraic metric with invariant foliation Λ, one can prove the following:

Theorem 2.3. *If $\Lambda^{\perp}$ is tangent to a geodesic of M at one point, then the geodesic is an integral manifold of $\Lambda^{\perp}$.*

In the context of the modified three-dimensional Turbiner conjecture, Λ is a rank 2 **G**-invariant distribution; thus, being a rank 1 distribution, $\Lambda^{\perp}$ is necessarily integrable. We then get:

Corollary 2.4. *If rank $(\Lambda^{\perp}) = 1$, then the integral curves of $\Lambda^{\perp}$ are geodesic trajectories.*

After an investigation of the possible foliations of $\mathbb{R}^3$ which are in accordance to our problem, we will be able to show that the partial separation will occur in Cartesian, cylindrical or spherical coordinates. Note that, as for the two-dimensional case, the extra hypotheses are necessary. Based on a primitive action, an explicit flat Lie algebraic Schrödinger operator, for which there is no separation of variable, will be exhibited at the end of this paper.

Let us now illustrate how the 2-imprimitivity of the action affects the operator constructed previously. Recall that if we consider the domain $\mathbf{R} \subset \mathbf{M_0}$ where the metric is positive definite, the operator reads as follows:

$$\mathcal{H}_0 = \Delta + \nabla(\alpha w + \beta \ln|v - u^2| + \gamma \ln|w - v|).$$

This operator, and its equivalent Schrödinger operator, illustrate clearly that separation arises from invariant foliations. Indeed the separation occurs in each of the three possible systems of coordinates. We shall not expect this in general. The three separations reflect the fact that the group action allows not only one but three distinct invariant foliations:

$$\{u = \text{const.}\}, \quad \{v = \text{const.}\}, \quad \{w = \text{const.}\}.$$

It is now guaranteed that $\mathcal{H}_0(\lambda) = f(\lambda)$ for $\lambda \in \{u, v, w\}$.

In terms of Cartesian coordinates (x, y, z), the original coordinates are given by

$$u = x, \qquad v = x^2 + y^2 \qquad w = x^2 + y^2 + z^2.$$

Thus the leaves of the foliation are planes, cylinders and spheres. For each of these foliations we will consider respectively the Cartesian, cylindrical (r, θ, z), and spherical (r, θ, ϕ) coordinates. Hence, in the Cartesian system, the coordinate x separates in the operator, and for the two other systems the radial coordinate can be separated. An other extra property of this operator is the fact that, for each of these coordinate systems, the operator also separates in the two other coordinate systems.

In these three coordinate systems, the operator is given by

$$\begin{aligned}
\mathcal{H}_0 &= \Delta + \nabla(\alpha(x^2 + y^2 + z^2) + \beta \ln|y| + \gamma \ln|z|),\\
\mathcal{H}_0 &= \Delta + \nabla(\alpha(r^2 + z^2) + \beta(\ln|r| + \ln|\sin\theta|) + \gamma \ln|z|),\\
\mathcal{H}_0 &= \Delta + \nabla(\alpha(r^2) + \beta(\ln|r| + \ln|\sin\phi| + \ln|\sin\theta|) + \gamma(\ln|r| + \ln|\cos\theta|)).
\end{aligned}$$

By applying the operator $\mathcal{H}_0$ to $\Phi(x_1, x_2, x_3) = \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)$, one easily verifies that the equation separates into three equations, each of them involving only one variable. After the required gauge transformation, the Schrödinger operator reads as

$$\begin{aligned}
\mathcal{H} &= \Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2(x^2 + y^2 + z^2) + \frac{\beta(\beta-1)}{y^2} + \frac{\gamma(\gamma-1)}{z^2},\\
\mathcal{H} &= \Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2(r^2 + z^2) + \frac{1}{r^2}\left[\frac{\beta(\beta-1)}{\sin^2\theta}\right] + \frac{\gamma(\gamma-1)}{z^2},\\
\mathcal{H} &= \Delta + \alpha(2\beta + 2\gamma + 3) + \alpha^2(r^2) + \frac{1}{r^2\sin^2\theta}\left[\frac{\beta(\beta-1)}{\sin^2\phi}\right] + \frac{1}{r^2}\left[\frac{\gamma(\gamma-1)}{\cos^2\theta}\right].
\end{aligned}$$

Once again, the three operators separate in their respective coordinate systems since the three potentials satisfy the separation condition; see [8] for more details.

The aim of the following sections is to prove the three-dimensional version of the following modified Turbiner's conjecture proved by Rob Milson in [7]:

Theorem 2.5. *Let $\mathcal{H}_0$ be a second-order Lie algebraic operator generated by the T_a as per (2.2), $g^{(ij)}$ the induced contravariant metric and* **R** *a connected component of* $\mathbf{M_0}$ *for which $g^{(ij)}$ is positive definite. Suppose the following statements are true:*

(1) *$\mathcal{H}_0$ is gauge equivalent to a Schrödinger operator;*
(2) *$(\mathbf{R}, g^{(ij)})$ is isometric to a subset of the Euclidean plane;*
(3) *the operators $\{T_a : a \in \mathfrak{g}\}$ act imprimitively;*
(4) **R** *is either compact, or can be compactified in such a way that the* **G***-action on* **R** *extends to a real–analytic action on the compactification.*

Then, both the eigenvalue equation $\mathcal{H}_0\psi = E\psi$, and the corresponding Schrödinger equation separate in either a Cartesian, or a polar coordinate system.

To this end, we will follow a path which is similar to the one followed by Rob Milson. However, as mentioned above, the extra requirement that the metric is diagonalizable and generic will be required.

3. Trapping and tiling

The main objective of this section is to show that, under some conditions, there is a global map from the Euclidean space to the positive definite region of a flat three-dimensional almost-Riemannian compact manifold. As for the planar case, the 3D-Tiling Theorem will follow principally from the 3D-Trapping Theorem. That latter assures that the flow of a gradient vector field can never cross the locus of degeneracy. Note that through this section, the contravariant metric will be taken to be diagonal and the genericity property will be needed to prove both theorems.

3.1. 3D-Trapping Theorem

The trapping property is a feature shared by every flat diagonal generic almost-Riemannian metric whose coefficients are analytic functions. All the work involved in the proof is based on an appropriate expression for the diagonal components of the Ricci curvature tensor. We will use the local coordinates (x, y, z), that we will sometimes denote as (x^1, x^2, x^3) to ease the notation. Thus, we define $H^i = \sum_j g^{ij} \frac{\partial}{\partial x^i} = g^{ii} \frac{\partial}{\partial x^i}$ and evaluate the diagonal components of the Ricci curvature tensor using the frame $\{H^1, H^2, H^3\}$. After some work of simplifications and rearrangements, we obtain the following three expressions:

$$\begin{aligned}
2(R_{11})g^2 &= -3(H^1(g))^2 + 2g(H^1(H^1(g))) + g^2[P_yQ_y + P_zR_z + 2QP_{yy} + 2RP_{zz} + P_x^2 - 2PP_{xx}] \\
&\quad + g[2P^3Q_xR_x + P^2QP_xR_x + P^2RP_xQ_x - PQ^2P_yR_y - PR^2P_zQ_z - 3QR^2P_z^2 - 3Q^2RP_y^2], \\
2(R_{22})g^2 &= -3(H^2(g))^2 + 2g(H^2(H^2(g))) + g^2[P_xQ_x + Q_zR_z + 2RQ_{zz} + 2PQ_{xx} + Q_y^2 - 2QQ_{yy}] \\
&\quad + g[2Q^3P_yR_y + PQ^2Q_yR_y + Q^2RP_yQ_y - P^2QR_xQ_x - QR^2P_zQ_z - 3PR^2Q_z^2 - 3P^2RQ_x^2], \\
2(R_{33})g^2 &= -3(H^3(g))^2 + 2g(H^3(H^3(g))) + g^2[Q_yR_y + P_xR_x + 2QR_{yy} + 2PR_{xx} + R_z^2 - 2RR_{zz}] \\
&\quad + g[2R^3P_zQ_z + QR^2P_zR_z + PR^2Q_zR_z - Q^2RP_yR_y - P^2RQ_xR_x - 3PQ^2R_y^2 - 3P^2QR_x^2].
\end{aligned}$$

Proposition 3.1. *Let g^{ij} be a diagonal, generic three-dimensional contravariant metric tensor with analytic coefficients. If the diagonal elements of the Ricci curvature tensor are identically zero, then there exist locally defined, analytic functions μ^1, μ^2 and μ^3 such that*

$$H^i(g) = \mu^i \cdot g \quad \text{for } i = 1, 2, 3.$$

Proof. Obviously, such functions exist around points where the determinant does not vanish. We can assume that g is zero at the origin and we focus on H^1 first. The ring of convergent power series with complex coefficients is a unique factorization domain; thus, up to multiplication by invertible functions, g factors uniquely into a product of irreducible, complex valued, analytic functions that are zero at the origin. Let f be such factor, and let k be its multiplicity, i.e. $g = f^k\sigma$, with f and σ coprime. Since $g^{(ij)}$ is generic, f^k is only a factor of one of the diagonal elements and if k is greater than one, f^{k-1} divides the three partial derivatives of this component. Thus, one easily sees that f^{2k-1} is a factor of the last two summands of $2(R_{11})g^2$:

$$g^2[P_yQ_y + \cdots - 2PP_{xx}] + g[2P^3Q_xR_x + \cdots - 3Q^2RP_y^2].$$

Since R_{11} is identically zero, the remaining summand, $3(H^1(g))^2 - 2g(H^1(H^1(g)))$, must also be divisible by f^{2k-1}. But, the preceding term can be written as

$$k(k+2)\sigma^2(H^1(f))^2 f^{2k-2} + \rho f^{2k-1}$$

where ρ is some analytic function. Thus $k(k+2)\sigma^2(H^1(f))^2$ must be divisible by f. Recall that σ is relatively prime to f and $k(k+2) > 0$, which forces $H^1(f)$ to be divisible by f. The same must be true for all non-invertible irreducible factors of g (and obviously true for the invertible factors); therefore $H^1(g)$ is divisible by g. The same argument holds for H^2 and H^3. □

Note that, without the genericity requirement, the first term of the last summand of $2(R_{11})g^2$ is only guaranteed to be divisible by f^{2k-2} which does not allow us to establish the claim. However, maybe another rearrangement of the terms could lead to the same conclusion without this extra hypothesis.

From this proposition, the 3D-Trapping Theorem follows immediately. As it is identical to the one given as Corollary 6.4.2 of [6], the proof is omitted.

Theorem 3.2 (*The 3D-Trapping Theorem*). *Let g^{ij} be as in the preceding theorem, and let f be an analytic function. Then the flow of $\nabla(f)$ can never cross the locus of degeneracy. More precisely, the trajectories of the flow of $\nabla(f)$ are either contained in the locus of degeneracy of g^{ij}, or never intersect it.*

3.2. 3D-Tiling Theorem

In what follows, using the 3D-Trapping Theorem, we will prove a three-dimensional version of Rob Milson's Tiling Theorem. As before, **M** is a compact, three-dimensional, almost-Riemannian manifold endowed with $g^{(ij)}$ a generic and flat metric with diagonal analytic coefficients. **R** is a region where the metric is positive definite.

The key argument for this proof is that either the degenerate points are unreachable, or the metric $g^{(ij)}$ is the pushforward of a non-degenerate metric $\tilde{g}^{(ij)}$. But before proving this proposition, the two following lemmas, deduced from the Proposition 3.1 and the genericity property of the metric, will simplify the subsequent work. Under the same hypothesis, we have the following:

Lemma 3.3. *If f is a non-invertible, irreducible factor of g^{ii}, then, for $i \neq j$, f is a factor of $g^{ii}_{x^j}$ and a factor of f_{x^j}.*

Proof. Suppose that $P = f^k\sigma$, where f is a non-invertible irreducible factor and $(f, \sigma) = 1$. By the genericity property of the metric, f is also coprime to Q and R, and from Proposition 3.1,

$$H^2(g) = Q(P_y QR + PQ_y R + PQR_y) = \mu^2 \cdot PQR.$$

P being a factor of all but one summand of the middle term, $P_y Q^2 R$ has also to be divisible by f^k, forcing f^k to divide P_y. Furthermore,

$$P_y = \begin{cases} kf^{k-1} f_y\sigma + f^k\sigma_y & \text{if } k > 1, \\ f_y & \text{if } k = 1, \end{cases}$$

and thus f needs to be a factor of f_y. $\square$

Lemma 3.4. *Given g^{ii}, a diagonal component of the contravariant metric g^{ij}, its non-invertible factors are functions of the variable x^i only.*

Proof. Consider f, a non-invertible factor of R. From the analyticity requirement, f can be expressed locally in terms of the following convergent power series:

$$f = \sum_{i,j,k=0}^{\infty} f_{ijk} x^i y^j z^k, \quad \text{where } f_{000} = 0.$$

According to Lemma 3.3, $f_x = f \cdot h$, for h an analytic function. The Taylor series of f_x can therefore be given as a product of two series:

$$f_x = \sum_{i,j,k=0}^{\infty} i f_{ijk} x^{i-1} y^j z^k = \sum_{i,j,k=0}^{\infty} f_{ijk} x^i y^j z^k \cdot \sum_{a,b,c=0}^{\infty} h_{abc} x^a y^b z^c. \tag{3.1}$$

If we suppose that there exist positive integers i such that $f_{ijk} \neq 0$, we can fix (α, β, γ), the smallest triple (with respect to the lexicographic order) such that $f_{\alpha\beta\gamma} \neq 0$. The coefficient of the monomial $x^{\alpha-1} y^\beta z^\gamma$ is $\alpha f_{\alpha\beta\gamma}$, and according to (3.1), it can also be given by

$$\sum_{\substack{i+a=\alpha-1 \\ j+b=\beta \\ k+c=\gamma}} f_{ijk} \cdot h_{abc}.$$

But all the coefficients f_{ijk} are zero since $i = \alpha - 1 - a < \alpha$. Consequently $f_{\alpha\beta\gamma} = 0$, a contradiction. So, we have $f_{ijk} = 0$ for all $i \neq 0$, and the same argument is used to show that $f_{ijk} = 0$ for all $j \neq 0$. Therefore

$$f = \sum_{k=0}^{\infty} f_{00k} z^k = f(z). \quad \square$$

We can now prove the following strong criteria for the unreachability of a degenerate point. As for the rest of this paper, the degenerate point will be taken to be the origin.

Proposition 3.5. *If the order of one of the diagonal components g^{ii} is greater than 1, then the origin is unreachable.*

Proof. Without loss of generality, $R = z^l(1 + f(x, y, z))$ where $l > 1$. We will compare the metric $g^{(ij)}$ to

$$\tilde{g}^{(ij)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix},$$

which is a flat metric whose origin is known to be unreachable. We can write the contravariant metric as

$$g^{(ij)} = \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & z^2\tilde{R} \end{pmatrix},$$

where $P, Q, \tilde{R}$ are non-singular at the origin. We can find a neighbourhood N and an upper bound $K > 0$ such that $\sup_N\{P, Q, \tilde{R}\} \leq K$. If we consider the region $\mathbf{R} \cap N$, we must have

$$\langle v, v\rangle_g \geq \frac{1}{K}\langle v, v\rangle_{\tilde{g}},$$

for all tangent vectors v. Indeed

$$\langle v, v\rangle_g = \frac{v_1^2}{P} + \frac{v_2^2}{Q} + \frac{v_3^2}{z^2\tilde{R}} \geq \frac{v_1^2}{K} + \frac{v_2^2}{K} + \frac{v_3^2}{z^2K} = \frac{1}{K}\langle v, v\rangle_{\tilde{g}}.$$

The length functional on curves in the metric g is bounded below by $\frac{1}{K}$ times the length functional in the metric $\tilde{g}$. The origin, unreachable with respect to $\tilde{g}$, is therefore unreachable with respect to g as well. □

Corollary 3.6. *If the origin is reachable, then the order of P, Q and R is at most* 1.

This leads us, up to relabeling of the variables, to three possibilities:

$$g = PQR = z(1 + f(x, y, z)), \tag{3.2}$$
$$g = PQR = yz(1 + f(x, y, z)), \tag{3.3}$$
$$g = PQR = xyz(1 + f(x, y, z)), \tag{3.4}$$

that enable us to prove the following key lemma.

Proposition 3.7. *A degenerate point is either an unreachable point, or there exists a contravariant, non-degenerate metric tensor $\tilde{g}^{(ij)}$ with analytic coefficients defined on some neighbourhood $N \subset \mathbb{R}^3$ and an analytic map $\phi : N \to R$ such that $\phi_*(\tilde{g}) = g$.*

Proof. If the origin is reachable, we are in one of three previous possibilities, say the case (3.4). Since each diagonal component has order 1, from Lemma 3.4, we can write $P = 4x\tilde{P}$, $Q = 4y\tilde{Q}$ and $R = 4z\tilde{R}$ where $\tilde{P}, \tilde{Q}, \tilde{R}$ are invertible. We consider the analytic map given by

$$\phi_3 := \begin{cases} x = \xi^2 \\ y = \eta^2 \\ z = \mu^2, \end{cases}$$

and we take N, the domain of this map, to be a neighbourhood of the origin sufficiently small that the image of the map is contained in R. One easily verifies that, via this map, $g^{(ij)}$ is the pushforward of

$$\tilde{g}^{(ij)} = \begin{pmatrix} \tilde{P} & 0 & 0 \\ 0 & \tilde{Q} & 0 \\ 0 & 0 & \tilde{R} \end{pmatrix},$$

which is non-degenerate at the origin. The cases (3.2) and (3.3) are resolved the same way, by considering respectively the maps

$$\phi_1 := \begin{cases} x = \xi \\ y = \eta \\ z = \mu^2 \end{cases} \quad \text{and} \quad \phi_2 := \begin{cases} x = \xi \\ y = \eta^2 \\ z = \mu^2. \end{cases} \qquad \square \tag{3.5}$$

The maps ϕ_i will be called 2^ith-*fold maps*. The name reflects the fact that the (ξ, η, μ) space generically covers the (x, y, z) space in a 2^i-to-one relationship, the exception being the folding planes, $z = 0$ for ϕ_1, $y = z = 0$ for ϕ_2 and $x = y = z = 0$ for ϕ_3. With this key lemma in hand, we can now assert that the positive definite region of the almost-Riemannian manifold is isometric to the Euclidean space modulo a discrete group of isometric symmetries. The proof is identical to the one given in [6] for the two-dimensional case and is based on the fact that reachable degenerate points are, in a way, removable. The three-dimensional case is a little simpler since the only analytic maps we need to consider are the 2^ith-fold maps. For these reasons, we will omit the proof.

Theorem 3.8 (*The 3D-Tiling Theorem*). *Let M be a compact three-dimensional flat almost-Riemannian manifold with diagonal generic metric. Then, there exists a globally defined, real–analytic map $\psi : \mathbb{R}^3 \to M$ such that $g^{(ij)}$ is the pushforward of the Euclidean metric, and such that ψ covers all of* **R** *plus the reachable portions of its boundary. Furthermore,* **R** *is isometric to the quotient $\mathbb{R}^3/\Gamma$, where Γ is the group of isometries γ such that $\psi = \psi\gamma$.*

Note that since ψ is a 2^ith-fold map, the group of isometries is indeed the group of reflections along the folding planes.

4. Foliations

In this section, we intend to determine what are the possible rank 2 foliations of $\mathbb{R}^3$ that are perpendicular to straight lines. This intermediate result, used in conjunction with the 3D-Tiling Theorem, will be used to prove that the function λ, whose level sets are the leaves of the invariant foliation, is a coordinate of the Cartesian, the cylindrical or the spherical coordinate system.

The rank 2 leaves are complete on **M** but they may cross the reachable part of the degenerate locus $\mathbf{D}_g$. It is not clear a priori that the pullback of these leaves is also complete in $\mathbb{R}^3$. Indeed, the rank of these leaves may drop where the Jacobian of $\psi : \mathbb{R}^3 \to \mathbf{R} \subset \mathbf{M}$ is degenerate. To avoid confusion, we denote as $\mathbf{S}_g$ the degeneracy locus of the foliation in $\mathbb{R}^3$ and we have the inclusion $\mathbf{S}_g \subseteq \psi^{-1}(\mathbf{D}_g)$.

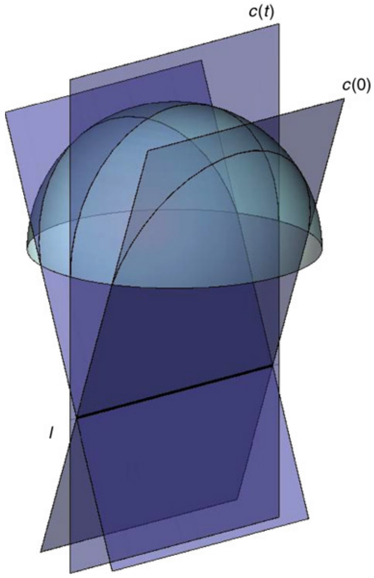
From the 3D-Tiling Theorem, **R**, the positive definite region of the manifold, is isometric to the quotient $\mathbb{R}^3/\Gamma$ where Γ is a discrete group of reflections. Thus $\mathbb{R}^3$ is tiled into isometric regions where the pullback of each leaf is repeated. If a rank 2 leaf of **R** crosses $\mathbf{D}_g$, its pullback will be reflected on the other side of $\psi^{-1}(\mathbf{D}_g)$. Hence the rank 2 leaves in $\mathbb{R}^3$ can be extended without restriction but they might fail to be smooth.

However, according to the 3D-Trapping Theorem, the trajectories of the flow of the gradient of λ are either contained in the locus of degeneracy, or never intersect it. This forces the leaves to cross $\psi^{-1}(\mathbf{D}_g)$ perpendicularly; thus, we can conclude that the rank 2 leaves are also smooth in $\mathbb{R}^3$.

Therefore throughout this section, Λ will denote a foliation of $\mathbb{R}^3$ which is of rank 2 almost everywhere. By degenerate points we refer to $\mathbf{S}_g$, the points where the rank drops. One easily sees that the rank 2 leaves never cross the locus of degeneracy. In accordance with Corollary 2.4, the leaves of $\Lambda^{\perp}$, the perpendicular foliation, are straight lines at every non-degenerate point. Our aim is to show that the leaves of Λ can only be planes, infinite cylinders or spheres. Before proving this result we need to establish some notation together with two lemmas.

For any point $x \in \mathbb{R}^3$, we denote as $\mathcal{M}_x$ its leaf and, for any curve c contained in a rank 2 leaf $\mathcal{M}$, we denote as S_c the ruled surface generated by the normals of $\mathcal{M}$ along c. Throughout this section the non-degenerate points will be dense and we will use the definitions found in [10] to describe solids.

Lemma 4.1. *Let $c(t)$ be a continuous family of curves parametrized by $t \in (-\delta, \delta)$, contained in $\mathcal{M}$, a rank 2 leaf. Suppose that for every $t_1 \neq t_2 \in (-\delta, \delta)$, the curves $c(t_1)$ and $c(t_2)$ are distinct almost everywhere. If all the surfaces $S_{c(t)}$ intersect each other, then they all intersect at $\mathcal{I}$ which is of dimension at most* 1.



Proof. Two different curves, $c(t_1)$ and $c(t_2)$, can only intersect at points; hence the two ruled surfaces, $S_{c(t_1)}$ and $S_{c(t_2)}$, can intersect in at most a one-dimensional set. For any $\tau \in (-\delta, \delta)$, we define $\mathcal{I}_\tau$ the intersection of $S_{c(\tau)}$ with all the other surfaces.

$$\mathcal{I}_\tau := \bigcup_{\substack{t\in(-\delta,\delta),\\ t\neq\tau}} S_{c(t)} \cap S_{c(\tau)} \neq \emptyset.$$

The set $\mathcal{I}_0$ cannot be a surface. Otherwise, by smoothness of the leaf, $\mathcal{I}_\rho$ would also be a surface for $|\rho| < \epsilon$ and ϵ sufficiently small. Thus we would have

$$\mathcal{I} := \bigcup_{|\rho|<\epsilon} \mathcal{I}_\rho,$$

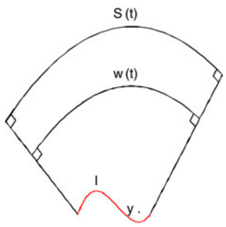
a three-dimensional degenerate set. Hence, every surface $S_{c(t)}$ will intersect $S_{c(0)}$ at $\mathcal{I}_0$ which is of dimension at most 1. The same argument holds for each set $\mathcal{I}_t$. Therefore, $\mathcal{I}_0 = \mathcal{I}_t$ for all $t \in (-\delta, \delta)$. □

Lemma 4.2. *Let $c(t)$ be a continuous family of curves parametrized by $t \in (-\delta, \delta)$ and contained in $\mathcal{M}$, a rank 2 leaf. If, for all $t \in (-\delta, \delta)$, each normal of $\mathcal{M}$ along $c(t)$ intersects a degenerate curve $\mathcal{I}$, then $\mathcal{I}$ is parallel to $c(t)$ for all $t \in (-\delta, \delta)$.*

Proof. Let $\mathcal{U} \subset \mathbb{R}^3$ be an open set contained in

$$S := \bigcup_{t\in(-\delta,\delta)} S_{c(t)}.$$

We pick $x \in \mathcal{U}$, a Λ-rank 2 point, and, by completeness, $\mathcal{M}_x$ is crossed perpendicularly by each normal associated with the family $c(t)$. This section of the leaf, given by $\bigcup_{t\in(-\delta,\delta)} \mathcal{M}_x \cap S_{c(t)}$, is therefore parallel to $\mathcal{M}$. We denote as $w(t)$ the intersection of $\mathcal{M}_x$ with $S_{c(t)}$ and we note that $c(t)$ is at constant distance from $w(t)$.

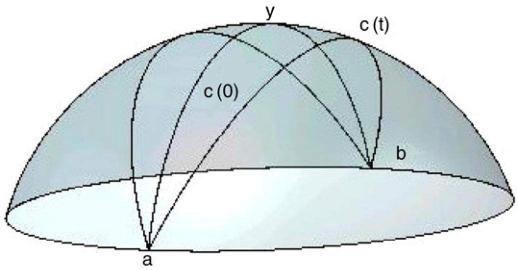


If $\mathcal{I}$ was not parallel to a curve $c(t)$, one could easily choose a rank 2 point, say y, sufficiently close to $\mathcal{I}$, for which $\mathcal{M}_y$ would intersect the degenerate set $\mathcal{I}$. This is impossible since the leaf $\mathcal{M}_y$ has Λ-rank 2 everywhere. □

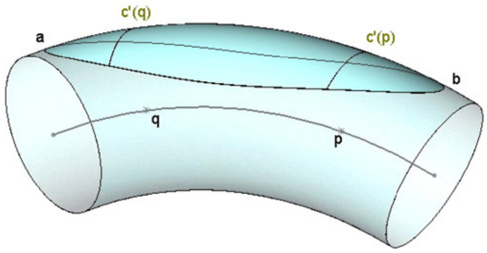
Remark that the intersection curve $\mathcal{I}$, being parallel to the leaves, has to be non-singular. Note also that the curves in the family are necessarily all parallel to each others. We can now prove the following three propositions, which, put together, will enable us to conclude about the three possible foliations.

Proposition 4.3. *Let $\mathcal{M}$ be a rank 2 leaf of the foliation Λ; if there exists an open $\mathcal{U} \subset \mathcal{M}$ for which the Gaussian curvature is positive, then $\mathcal{M}$ is a sphere.*

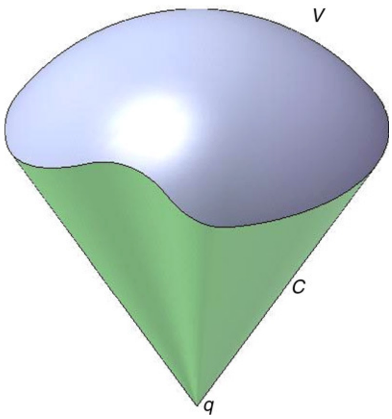
Proof. Let $y \in \mathcal{U}$ and consider $c(0) \subset \mathcal{U}$ the segment of the curve starting at y and following the leaf $\mathcal{M}$ in a given direction $\pm\vec{v}$. If we fix the end points a and b and slide $c(0)$ in the two directions perpendicular to $\vec{v}$, we get a family of curves $c(t)$, $t \in (-\delta, \delta)$, contained in $\mathcal{U}$. We denote as $\mathcal{V}$ the subset generated by the curves $c(t)$.



Since the two principal curvatures are non-zero in $\mathcal{U}$, the normal surfaces $S_{c(t)}$ intersect in a connected component and, by Lemma 4.1, the intersection is either a point q, or a connected curve $\mathcal{I}$, parallel to $c(t)$ by Lemma 4.2. Suppose first that the intersection is a curve. Being parallel to $\mathcal{I}$, the surface $\mathcal{V}$ has to be contained in a twisted cylinder centered at $\mathcal{I}$. For each point p on the curve $\mathcal{I}$, we denote as $c'(p)$ the intersection of $\mathcal{V}$ with the normal plane of $\mathcal{I}$ at p.

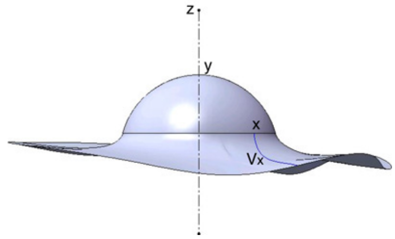


Again, $c'(p)$ is a continuous family of curves, here parametrized by p. Since $\mathcal{I}$ is parallel to the surface $\mathcal{V}$, the surface $S_{c'(p)}$ is contained in the normal plane of p. By the curvature hypothesis, these plane sections intersect, say in $\widetilde{\mathcal{I}}$, and by Lemma 4.1, $\widetilde{\mathcal{I}}$ is either a point, or a straight line. The latter is impossible since, by Lemma 4.2 the line segment $\widetilde{\mathcal{I}}$ would be parallel to every $c'(p)$ whose curvature is not zero. Therefore, $\widetilde{\mathcal{I}}$ has to be a point q, and all the normals of $\mathcal{V}$ intersect in q. Necessarily, the curve $\mathcal{I}$ needs to restrict to the point q. In that case $\mathcal{V}$ is parallel to q; hence $\mathcal{V}$ is contained in a sphere.



The Gaussian curvature on $\mathcal{V}$ has to be constant, say $\frac{1}{R^2}$, and all the points of $\mathcal{V}$ are at distance R from q. We are left with showing that if we extend $\mathcal{V}$ to the entire leaf $\mathcal{M}$, the distance between $\mathcal{M}$ and q will be preserved, i.e. $\mathcal{M}$ is a sphere.

Without loss of generality we consider $\mathcal{V}$ to be the maximal spherical cap with pole y. We denote as $\mathcal{C}$ the cone with apex q generated by the normals of $\mathcal{V}$, $\partial\mathcal{V}$ the boundary of $\mathcal{V}$, and $\mathcal{V}_x$ the curve obtained by extending $\mathcal{V}$ through $x \in \partial\mathcal{V}$ perpendicularly to $\partial\mathcal{V}$ along the leaf. We consider as the Z-axis the line containing q and y and we define as $\mathcal{P}_w$ the *alignment plane*, containing the Z-axis and the point w.



Remark that if the Gaussian curvature of $\mathcal{M}$ changes along the curve $\mathcal{V}_x$, by smoothness of the leaves, for $y \in \partial\mathcal{V}$ close to x, the curvature will also change along the curves $\mathcal{V}_y$. The key point here is that if there are changes in the curvature, the surfaces $S_{\mathcal{V}_x}$ have to eventually leave their alignment planes. Indeed we need to avoid two-dimensional intersection with $\mathcal{C}$; otherwise, taking this together with the intersections of the surfaces $S_{\mathcal{V}_y}$ with $\mathcal{C}$, we would get a three-dimensional degenerate set. We are left with two possibilities. Either $\mathcal{C} \cap S_{\mathcal{V}_x}$ is always q, which is impossible since the curvature changes, or the $S_{\mathcal{V}_x}$ eventually leave the cone, which forces the normals of $S_{\mathcal{V}_x}$ to leave their alignment plane. Note also that, if a normal stays in the alignment plane, it has to intersect $\mathcal{C}$ at q. The main objective now is to show that if we extend the spherical cap, the normal lines stay in their alignment planes, intersecting the cone at q and preserving the curvature of the spherical cap.

Let γ_ϵ be the closed curve in $\mathcal{M}$ which is at distance ϵ outside $\partial\mathcal{V}$. By completeness, such a curve always exists for ϵ sufficiently small, say $\epsilon < \varepsilon$. We want to show first that along such a path, all the normal lines swing in the same direction with respect to their alignment planes. Let $x \in \gamma_\epsilon$, and assume that the curve is traversed in the clockwise direction. Note that if the dot product between the normal line and the tangent vector of the curve γ_ϵ is positive, then there is an increase of the Z-value of γ_ϵ around x. Suppose we can take two curves, $\mathcal{V}_x$ and $\mathcal{V}_y$, for which the normal lines swing in different directions. Say, without loss of generality, that along each curve γ_ϵ, $\epsilon \in [0, \epsilon_1]$, either the direction changes only once, or the normal is in the alignment plane at $\mathcal{V}_x$ and then rolls in at most one direction. By smoothness of the leaf, for each γ_ϵ, there would be at least a point α_ϵ for which its normal lies in the alignment plane $\mathcal{P}_{\alpha_\epsilon}$, and hence intersects $\mathcal{C}$ at q. With an appropriate choice of α_ϵ, for every ϵ in $[0, \epsilon_1]$, we could generate the curve $\alpha(\epsilon)$ parallel to q. Since $\alpha(0) \in \partial\mathcal{V}$ the curve would be at distance R of q.

Necessarily, there should be another change of direction, say between $\mathcal{V}_y$ and $\mathcal{V}_z$, along the curves γ_ϵ for $\epsilon \in [0, \epsilon_2]$. We would then get another curve $\beta(\epsilon)$ at distance R from q. Hence for all $\epsilon \in [0, \tilde{\epsilon}]$, where $\tilde{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$, α_ϵ and β_ϵ would be at distance ϵ from $\partial\mathcal{V}$ and at distance R from q. Thus, for a fixed ϵ, they would necessarily have the same Z-value. But along γ_ϵ, everywhere in between $\alpha(\epsilon)$ and $\beta(\epsilon)$, the normal lines are on the same side of their alignment plane $\mathcal{P}_z$, implying a strict increase (or decrease) of the Z-value between the two points. This is impossible; hence the normal can only swing in one direction.

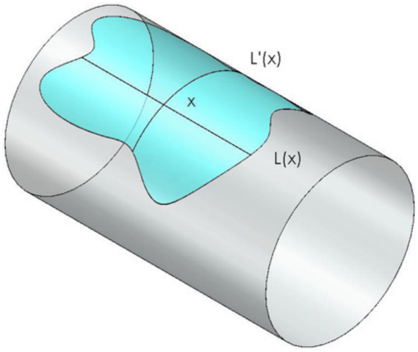
Therefore, the Z-value is monotonic as we follow the closed curve γ_ϵ in a given direction. This is impossible except if the Z-value is constant, that is, if the normals stay in their alignment planes. Thus, for $\epsilon < \varepsilon$ the normal lines of $\mathcal{M}$ along γ_ϵ must intersect the cone at q. Note that the curves $\mathcal{V}_x$ stay parallel to q when they intersect γ_ϵ. So we can increase $\mathcal{V}$ to

$$\widetilde{\mathcal{V}} := \mathcal{V} \bigcup_{\epsilon < \varepsilon} \gamma_\epsilon,$$

a bigger spherical cap. This contradicts the maximality of $\mathcal{V}$. Hence $\mathcal{V}$ has to be a sphere and is indeed the entire $\mathcal{M}$. □

Proposition 4.4. *Let $\mathcal{M}$ be a leaf of the foliation Λ; if there exists an open $\mathcal{U} \subset \mathcal{M}$ for which one of the principal curvatures is identically zero and the other is non-vanishing, then $\mathcal{M}$ is an infinite cylinder.*

Proof. Let $\mathcal{L}(x)$ and $\mathcal{L}'(x)$ be the principal curves passing through $x \in \mathcal{U}$ related respectively to the vanishing and the non-vanishing principal curvatures, say $0 \equiv \lambda_1 < \lambda_2$. Note that $\mathcal{L}(x)$ is a line. Since λ_2 is never vanishing on $\mathcal{U}$, the normal surfaces $S_{\mathcal{L}(x)}$ intersect, and from Lemmas 4.1 and 4.2, all the lines $\mathcal{L}(x)$ are parallel to $\mathcal{I}$, which has to be a line also. By following the leaf along $\mathcal{L}'(x)$, the distance between $\mathcal{I}$ and the points in $\mathcal{U}$, say R, has to be preserved. Therefore, $\mathcal{U}$ has to be contained in a radius R cylinder.

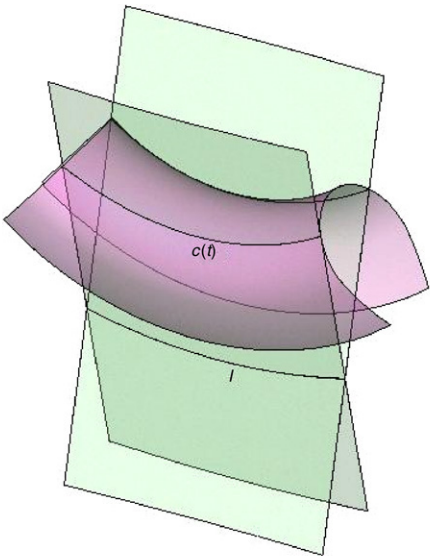


We can extend $\mathcal{L}'(x)$ outside $\mathcal{U}$, and for ϵ sufficiently small, we define the curve γ_ϵ to be the union of the points along these extensions that are at distance ϵ from the boundary of $\mathcal{U}$. The normal surfaces S_{γ_ϵ} have to intersect; otherwise it would create a three-dimensional degenerate set with the normal lines of $\mathcal{U}$. By Lemma 4.1, these normal surfaces need to intersect in a curve and since the leaf is smooth, this curve has to be $\mathcal{I}$. By Lemma 4.2, these curves are parallel to the line $\mathcal{I}$ so, by extending $\mathcal{L}'(x)$ along $\mathcal{M}$, we get a cylinder $\mathcal{C}$.

If we extend the principal curve $\mathcal{L}(x)$ outside the cylinder $\mathcal{C}$, the curve obtained, say $\mathcal{L}(x)^*$, needs to be a straight line. Otherwise, as for the previous case, it would create a three-dimensional degenerate set while intersecting the normal lines of the cylinder. Consequently, $\mathcal{M}$ has to be an infinite cylinder. □

Proposition 4.5. *If $\mathcal{M}$ is a leaf of the foliation Λ, then there is no open $\mathcal{U} \subset \mathcal{M}$ for which $\lambda_1 \cdot \lambda_2 < 0$.*

Proof. Assume the opposite, and pick $y \in \mathcal{U}$. We consider $c(0)$, the intersection of $\mathcal{U}$ with the principal curve through y associated with $\lambda_1 < 0$. We denote as $c(t)$ the curves of $\mathcal{U}$ parallel to $c(0)$. By the curvature hypothesis, the normal surfaces $S_{c(t)}$, intersect. Hence by Lemmas 4.1 and 4.2, they intersect at $\mathcal{I}$, a line parallel to the curves $c(t)$.



After crossing the intersection curve, the normal lines of $\mathcal{U}$ will cross leaves for which the two principal curvatures have the same sign; hence by Proposition 4.3, the leaves on the other side of $\mathcal{I}$ will be spheres. Such cohabitation of a hyperbolic surface and spheres is impossible for the foliation Λ. □

Corollary 4.6. *The non-degenerate leaves of the foliation Λ are planes or cylinders or spheres.*

There seems to be a deep connection between the leaves arising from a 2-imprimitive action and *isoparametric manifolds*. Recall that a hypersurface $\mathbf{M}^n$ of a Riemannian manifold $\mathbf{V}^{n+1}$ is an isoparametric manifold if $\mathbf{M}^n$ is locally a regular level set of a function λ with the property that both $\|\nabla(\lambda)\|$ and $\Delta(\lambda)$ are constant on the level sets of λ. One easily check that the three possible leaves obtained in this section are indeed isoparametric manifolds. The interesting point is that the only complete isoparametric hypersurfaces of $\mathbb{R}^3$ are planes, spheres and round cylinders and this classification holds for any hypersurfaces of $\mathbb{R}^{n+1}$; see [4] for details. Hence, the theory of isoparametric manifolds should provide a good setting to approach Turbiner's conjecture in higher dimensions.

5. 3D-Turbiner's conjecture

We have now all the tools needed to prove the 3D modified Turbiner's conjecture. This main theorem is a partial affirmation of Turbiner's conjecture in three dimensions. First note that the original conjecture involved complete separation while here we succeed in proving that the equations separate partially. By a partial separation, we mean that the equations separate into two equations, one involving only one variable, the other involving the remaining variables. Also, three assumptions need to be added to the original conjecture: the underlying action has to be 2-imprimitive, the manifold on which the operator is defined has to be compact or can be compactified and the contravariant metric needs to be diagonal and generic. As for the two-dimensional case, the recipe is to pull back the invariant foliation to the Euclidean environment where the leaves can only be prescribed surfaces. Then working out the formulas for the operators in the appropriate coordinates, one succeeds in isolating one of the variables. The major differences from the two-dimensional case are: the necessity of the generic requirement and the partial separation obtained. As mentioned previously, at least one of the extra requirements of the modified version of the conjecture is necessary; a counter-example will be given in the next section.

Theorem 5.1 (*3D Modified Turbiner's Conjecture*). *Let $\mathcal{H}_0$ be a second-order Lie algebraic operator generated by the operators T_a as per (2.2), $g^{(ij)}$ be the induced contravariant metric and* **R** *be a connected component of* $\mathbf{M_0}$ *for which $g^{(ij)}$ is positive definite. Suppose the following statements are true:*

(1) *$\mathcal{H}_0$ is gauge equivalent to a Schrödinger operator;*
(2) *$(\mathbf{R}, g^{(ij)})$ is flat;*
(3) *the operators $\{T_a : a \in \mathfrak{g}\}$ act 2-imprimitively ;*
(4) **R** *is either compact, or can be compactified in such a way that the* **G***-action on* **R** *extends to a real–analytic action on the compactification;*
(5) *the metric $g^{(ij)}$ is diagonalizable and generic or* **M** *is a transverse, type changing manifold.*

Then, both the eigenvalue equation $\mathcal{H}_0\psi = E\psi$, and the corresponding Schrödinger equation separate partially in a Cartesian, cylindrical or spherical coordinate system.

Proof. We denote as Λ the T_a-invariant foliation. The leaves are the level sets of a function, say λ, and, from Proposition 2.2, this foliation is also **G**-invariant. The almost-Riemannian manifold $(\mathbf{R}, g^{(ij)})$ fulfills the hypothesis of the Tiling Theorem; thus there exists a real–analytic map $\Phi : \mathbb{R}^3 \to \mathbf{R}$ for which $g^{(ij)}$ is the pushforward of the Euclidean metric. It is then possible to pull back the rank 2 foliation Λ to get $\Phi^*(\Lambda)$ which is of rank 2 almost everywhere. From Corollary 2.4, $\Phi^*(\Lambda)$ is locally orthogonal to a foliation by geodesics that are, in this context, straight lines. The rank 2 leaves are complete; hence, we can apply Corollary 4.6, to conclude that there exist Cartesian coordinates (x, y, z) such that the leaves are given by the level sets of λ, where λ is x, $x^2 + y^2$ or $x^2 + y^2 + z^2$.

We will now move the setting to $\mathbb{R}^3$. There is still the local action of the group **G**, but this action is non-degenerate only whenever the Jacobian of Φ is not degenerate. Separation is a local phenomenon, so for the present purpose we can safely ignore the point of degeneracy.

The operator $\mathcal{H}_0$ is gauge equivalent to a Schrödinger operator $\mathcal{H}$; hence it must satisfy the closure condition. That means that there exists a function σ such that

$$\mathcal{H}_0 = \Delta + \nabla(\sigma) + V_0.$$

From the 2-imprimitivity of the action, $\mathcal{H}_0(\lambda) = f(\lambda)$ and one easily verifies that the Laplacian of λ is a function of λ for the three possible coordinate systems. Thus Λ is also invariant with respect to $\nabla(\sigma) + V_0$. But, remark that

$$[\nabla(\sigma) + V_0](\lambda^2) - \lambda[\nabla(\sigma) + V_0](\lambda) = \lambda\nabla(\sigma)(\lambda),$$

which forces both $\nabla(\sigma)$ and V_0 to be functions of λ. Depending of the metric, one easily check that this forces the gauge factor to separate in the following way:

$$\sigma(x, y, z) = \rho(x) + \eta(y, z),$$
$$\sigma(r, \theta, z) = \rho(r) + \eta(\theta, z),$$
$$\sigma(r, \theta, \phi) = \rho(r) + \eta(\theta, \phi).$$

Therefore the equation $\mathcal{H}_0\psi = E\psi$ separates partially and we are left to show that the Schrödinger equation also separates.

Recall that V, the potential of the Schrödinger operator, is given by

$$V = V_0 + \nabla(\sigma)^2 + \Delta(\sigma), \quad \text{where } V_0 = V_0(\lambda).$$

After easy computations, the potentials are given respectively by

$$V = F(x) + G(y, z),$$
$$V = F(r) + \frac{1}{r^2}G(\theta, z) + H(\theta, z),$$
$$V = F(r) + \frac{1}{r^2}G(\theta, \phi),$$

where F depends on the two functions ρ and V_0, while G and H depend on η.

This is sufficient for concluding that the Schrödinger equation

$$(\Delta + V)\Psi = E\Psi$$

separates partially in Cartesian, cylindrical or spherical coordinates. Indeed, we can perform respectively the following separations:

$$[\partial_{xx} + F(x) - E]\Psi_1(x) = \alpha\Psi_1(x)$$
$$[\partial_{yy} + \partial_{zz} + G(y, z)]\Psi_2(y, z) = -\alpha\Psi_2(y, z),$$
$$\left[\partial_{rr} + \frac{1}{r}\partial_r + F(r) - E\right]\Psi_1(r) = \left(\frac{1}{r^2}\alpha + \beta\right)\Psi_1(r)$$
$$[\partial_{\theta\theta} + \partial_{zz} + G(\theta, z) + H(\theta, z)]\Psi_2(\theta, z) = -(\alpha + \beta)\Psi_2(\theta, z),$$
$$\left[\partial_{rr} + \frac{2}{r}\partial_r + F(r) - E\right]\Psi_1(r) = \frac{1}{r^2}\alpha\Psi_1(r)$$
$$\left[\frac{1}{\sin^2\phi}\partial_{\theta\theta} + \partial_{\phi\phi} + \cot\phi\partial_\phi + G(\theta, \phi)\right] = -\alpha\Psi_2(\theta, \phi),$$

where α and β are separation constants. □

6. Counter-example

To conclude, we exhibit an example to show that the extra hypotheses cannot be omitted. Indeed, we construct a Lie algebraic Schrödinger operator using generating operators that act in a primitive way and we check that the potential cannot be separated, even partially. This counter-example is the natural generalization of the one given in [7] for the two-dimensional case. It also motivates the notion of an almost-Riemannian manifold by realizing the quotient of Euclidean space by an infinite reflection group. The general idea for this type of construction is to find a set of basic invariants and use them as coordinates.

This construction is a bit different from the usual one. Instead of choosing first the coefficients that generate a Lie algebraic operator and then verifying afterwards the closure condition, we proceed in a different order. We first create an almost-Riemannian manifold intimately related to the Lie algebra, then create an operator satisfying the closure condition and finally check whether there is a choice of coefficients that generate that operator. We consider in this example the Lie algebra $\mathfrak{sl}_4$, $\mathfrak{h}$ its diagonal Cartan subalgebra equipped with the usual Killing inner product and W, the affine Weil group associated with the root system. We denote as L_1, L_2, L_3 and L_4 the weights associated with the diagonal entries of a trace-free diagonal matrix, where $L_4 = -L_1 - L_2 - L_3$. Taking L_1, L_2 and L_3 as non-orthogonal coordinates, the contravariant form of the metric tensor is given, in an appropriate basis, by

$$\begin{pmatrix} 2 & -2/3 & -2/3 \\ -2/3 & 2 & -2/3 \\ -2/3 & -2/3 & 2 \end{pmatrix}.$$

We define $z_k = \mathrm{e}^{2\pi \mathrm{i} L_k}$, the generators of the corresponding torus of diagonal unimodular matrices. The algebra of W-invariant elements of the complexified coordinate ring is generated by χ_1, χ_2 and χ_3, the characters of the three fundamental representations of $\mathfrak{sl}_4\mathbb{C}$; see [1] for more details. These three invariants are given by

$$\begin{aligned}
\chi_1 &= z_1 + z_2 + z_3 + z_4, \\
\chi_2 &= z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4, \\
\chi_3 &= z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4,
\end{aligned}$$

and one easily compute the contravariant metric associated with this algebra:

$$g^{(ij)} = -8\pi^2 \begin{pmatrix} \chi_1^2 - 8/3\chi_2 & \frac{2}{3}(\chi_1\chi_2 - 6\chi_3) & \frac{1}{3}(\chi_1\chi_3 - 16) \\ \frac{2}{3}(\chi_1\chi_2 - 6\chi_3) & \frac{4}{3}(\chi_2^2 - 2\chi_1\chi_3 - 4) & \frac{2}{3}(\chi_2\chi_3 - 6\chi_1) \\ \frac{1}{3}(\chi_1\chi_3 - 16) & \frac{2}{3}(\chi_2\chi_3 - 6\chi_1) & \chi_3^2 - 8/3\chi_2 \end{pmatrix}.$$

On the real torus, χ_1 and χ_3 are complex conjugates, while χ_2 is real valued. Thus, fundamental invariants, denoted as (x, y, z), are given by the real and imaginary parts of χ_1 and by χ_2. In the real coordinates, the corresponding contravariant metric $g^{(ij)}$, modulo a factor $\frac{-8\pi^2}{3}$, reads as follows:

$$\begin{pmatrix} 2x^2 - z^2 - 4y - 8 & 2(xy - 6y) & 3xz \\ 2(xy - 6y) & 4(y^2 - 2x^2 - 2z^2 - 4) & 2(yz + 6z) \\ 3xz & 2(yz + 6z) & 2z^2 - x^2 + 4y - 8 \end{pmatrix}. \tag{6.1}$$

For convenience, we will omit this $-8\pi^2/3$ factor and one can verify that the Riemannian curvature tensor is identically zero where the metric is positive definite. The locus of degeneracy of the metric is given by

$$\begin{aligned}
\sigma = & -16(x^2 + z^2)^3 + (x^2 + z^2 + 58/39)(320y^2 + 768) \\
& + (x^2 - z^2)(32y^3 - 1152y) + (x^4 + z^4 - 352/39)(-4y^2 + 240) \\
& - 144(x^4 - z^4)y - 8x^2y^2z^2 - 1248x^2z^2 - 64y^4 = 0.
\end{aligned}$$

The objective now is to construct a Lie algebraic Schrödinger operator on a space for which the contravariant metric is given by (6.1). The entries of the matrix are degree 2 polynomials; hence the metric tensor can be generated by $\mathfrak{a}_3$, the Lie algebra of infinitesimal affine transformations of $\mathbb{R}^3$. A set of generators of $\mathfrak{a}_3$ is given by

$$\begin{aligned}
&T_1 = \partial_x, \quad T_2 = \partial_y, \quad T_3 = \partial_z, \quad T_4 = x\partial_x, \quad T_5 = x\partial_y, \quad T_6 = x\partial_z, \\
&T_7 = y\partial_x, \quad T_8 = y\partial_y, \quad T_9 = y\partial_z, \quad T_{10} = z\partial_x, \quad T_{11} = z\partial_y, \quad T_{12} = z\partial_z,
\end{aligned}$$

and one easily sees that there is no function λ for which λ and $T_\alpha(\lambda)$ are functionally dependent for all the generators T_α. Thus, these operators do not admit an invariant foliation and the realization of $\mathfrak{a}_3$ is therefore primitive. In terms of these operators, the Laplacian, in the (x, y, z) coordinates, is given by

$$\begin{aligned}
\Delta = & -8T_1^2 - 16T_2^2 - 8T_3^2 + 2T_4^2 - 8T_5^2 - T_6^2 + 4T_8^2 - T_{10}^2 - 8T_{11}^2 \\
& - 2\{T_1, T_7\} + 2\{T_3, T_9\} - 12\{T_2, T_4\} + 12\{T_2, T_{12}\} + 3\{T_4, T_{12}\}
\end{aligned}$$

$$+2\{T_4, T_8\} + 2\{T_8, T_{12}\} - 2T_4 - 4T_8 - 3T_{12}.$$

For σ the determinant of the contravariant metric (6.1), one easily verifies that

$$\nabla \log \sigma = 12(T_4 + T_{12}) + 16T_8. \tag{6.2}$$

Therefore, the operator

$$\mathcal{H}_0 = -\Delta + \nabla \log \sigma$$

is Lie algebraic and gauge equivalent to $\mathcal{H}$, a Schrödinger operator, via the gauge transformation:

$$\mathcal{H} = \mathrm{e}^{-\log(\sigma)/2} \circ \mathcal{H}_0 \circ \mathrm{e}^{\log(\sigma)/2} = -\Delta + U.$$

The potential U can be computed,

$$\begin{aligned} U = {} & 80 - 64[x^6 + 3x^4z^2 + 3x^2z^4 + z^6 - 18x^4y + 2x^2y^3 - 2y^3z^2 + 18yz^4 \\ & + 60x^4 + 60x^2y^2 - 312x^2z^2 - 8y^4 + 60y^2z^2 + 60z^4 + \\ & - 360x^2y + 360yz^2 + 336x^2 + 192y^2 + 336z^2 - 640]\sigma^{-1}, \end{aligned}$$

and can also be described in terms of the affine coordinates (L_1, L_2, L_3) by

$$U = 80 + \sum_{1 \le j < k \le 4} \frac{1}{\sin^2(\pi i (L_j - L_k))}. \tag{6.3}$$

To conclude our counter-example, we need to show that the Schrödinger equation $\mathcal{H}$ cannot be solved, even partially, by separation of variables. The potential here is symmetrical in the three variables; hence the separation with respect to one variable would imply a separation in the others, and thus a complete separation of variables. The Schrödinger equation can be solved by separation of variables in only eleven coordinate systems, nine of which (with the exception of paraboloidal coordinates) are particular cases of the ellipsoidal coordinates. According to [8], these coordinates are: rectangular (Cartesian), circular cylinder, elliptic cylinder, parabolic cylinder, spherical, conical, parabolic, prolate spheroidal, oblate spheroidal, paraboloidal, ellipsoidal coordinates.

An appropriate change of coordinates gives the orthonormal system (y_1, y_2, y_3) and one gets the following similar potential:

$$U = 80 + \sum_{1 \le j < k \le 3} \frac{1}{\sin^2(2\sqrt{2/3}\pi i (y_j \pm y_k))} \tag{6.4}$$

Since the first nine coordinate systems are particular cases of the last one, we only have to show that there is no separation possible in the last two coordinate systems: ellipsoidal and paraboloidal.

The ellipsoidal system of coordinates (ξ_1, ξ_2, ξ_3) is related to the Cartesian one by

$$\begin{aligned} y_1 &= \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}}, \\ y_2 &= \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}}, \\ y_3 &= \frac{\xi_1\xi_2\xi_3}{ab}, \quad \text{where } \xi_1^2 \ge a^2 \ge \xi_2^2 \ge b^2 \ge \xi_3^2 \ge 0, \end{aligned}$$

while, for the paraboloidal system, we have

$$\begin{aligned} y_1 &= \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{(a^2 - b^2)}}, \\ y_2 &= \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{(b^2 - a^2)}}, \end{aligned}$$

$$y_3 = \sqrt{\frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 - a^2 - b^2)}, \quad \text{where } \xi_1^2 \geq a^2 \geq \xi_2^2 \geq b^2 \geq \xi_3^2 \geq 0.$$

For these two systems, a given potential U separates if and only if it is of the form

$$U = \frac{(\xi_2^2 - \xi_3^2)U_1(\xi_1) + (\xi_1^2 - \xi_3^2)U_2(\xi_2) + (\xi_1^2 - \xi_2^2)U_3(\xi_3)}{(\xi_1^2 - \xi_2^2)(\xi_2^2 - \xi_3^2)(\xi_1^2 - \xi_3^2)}.$$

After the suitable substitution, the potential U, in terms of the new coordinates (ξ_1, ξ_2, ξ_3), fails to be of that required form. Therefore, no separation is possible.

Thus, this example emphasizes the necessity of the extra hypotheses that we needed to add to the original conjecture. Here, at least one of these hypotheses, the imprimitivity of the action, fails to be satisfied and the Schrödinger equation cannot be solved by separation of variables, even partially.

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